Orbital Mechanics Elliptic Orbits

Sasan Ardalan, Ph.D.

2024

Contents

List of Figures

Chapter 1

Introduction

Elliptic orbits are covered with derivations and also Mathematica® code to illustrate the velocity vector as, for example, a satellite orbits the Earth. Also covered, with Mathematica® code included, is the position of a satellite with specified time (time of flight).

Chapter 2

The Ellipse and Key Equations

2.1 Geometry

Figure 1 shows the geonmetry of an ellipse. It is important to grasp the relationships between the various parameters in an ellipse.

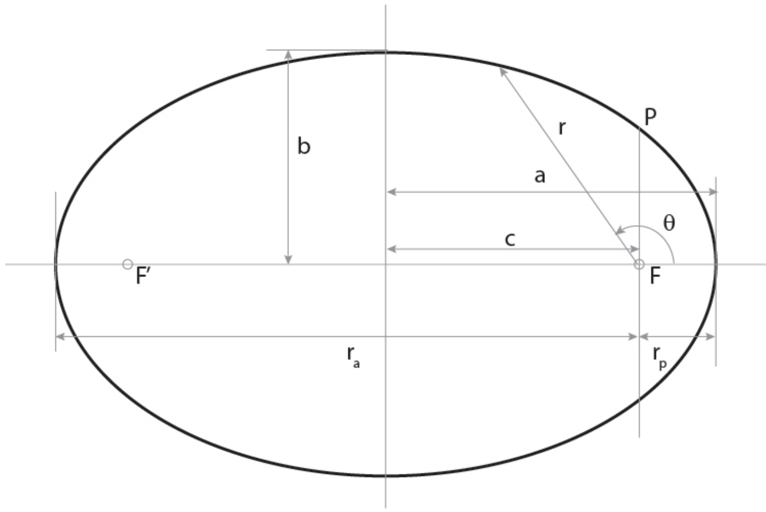

Figure 1: Ellipse

To start, note that the equation for an ellipse in polar coordinates is:

$$r = \frac{p}{1 + e\cos\theta} \tag{1}$$

where e is the eccentricity and is defined as:

$$e = \frac{c}{a} \tag{2}$$

Now from Figure 1:

$$a = \frac{r_p + r_a}{2} \tag{3}$$

Also

$$c = \frac{r_a - r_p}{2} \tag{4}$$

From Figure 1, setting $\theta = 0$,

$$r_p = \frac{p}{1 + e} \tag{5}$$

and setting $\theta = \pi$,

$$r_a = \frac{p}{1-e} \tag{6}$$

So,

$$a = \frac{r_p + r_a}{2} = \frac{1}{2}p(\frac{1}{1+e} + \frac{1}{1-e}) = \frac{p}{1-e^2} \tag{7}$$

Or,

$$p = \frac{a}{1-e^2} \tag{8}$$

Now lets relate e to a and b. Recall,

$$e = \frac{c}{a} \tag{9}$$

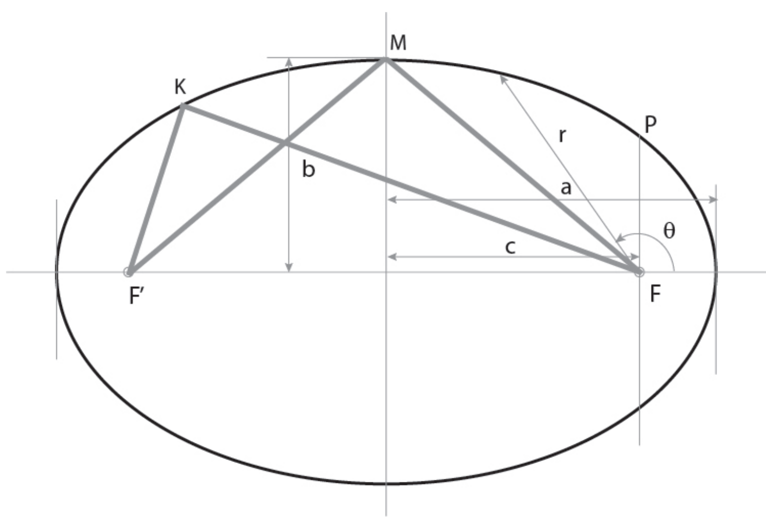

Figure 2: Ellipse Definition

From Figure 2, which is the definition of an ellipse, the two sums hold:

$$F'K + FK = 2a \tag{10}$$

and

$$F'M + FM = 2a \tag{11}$$

Subsequently ,

$$FM = F'M = a \tag{12}$$

Therefore,

$$a^2 = b^2 + c^2 \tag{13}$$

So that,

$$e = \frac{\sqrt{a^2 - b^2}}{a} \tag{14}$$

Thus the eccentriciy has been related to the semimajor axis a and the semiminor axis b.

2.2 Summary of Equations

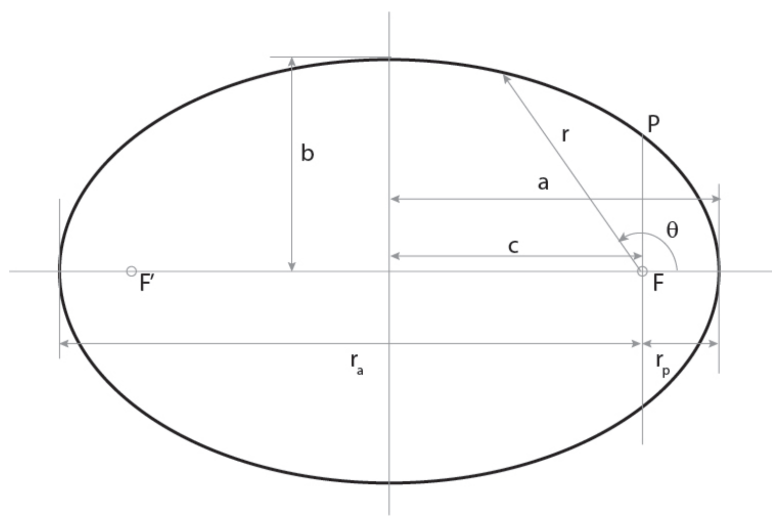

Figure 3: Ellipse Reference

$$\boxed{\begin{aligned} e &= \frac{c}{a} && (15a) \\ e &= \frac{\sqrt{a^2 - b^2}}{a} && (15b) \\ a &= \frac{r_p + r_a}{2} && (15c) \\ c &= \frac{r_p - r_a}{2} && (15d) \\ b &= \sqrt{a^2 - c^2} && (15e) \\ b &= a\sqrt{1 - e^2} && (15f) \\ p &= a(1 - e^2) && (15g) \\ r_p &= \frac{p}{1 + e} && (15h) \\ r_a &= \frac{p}{1 - e} && (15i) \\ r &= \frac{p}{1 + e\cos\theta} && (15j) \end{aligned}}$$

Chapter 3

Derivation of Equations for Elliptic Orbits

The equations governing the motion of an object of mass m and an object of mass M will be derived. The results of this chapter are summerized in the following chapters for reference. The derivation of the equations of motion are not trivial. We have included every single step with explanations and illustrations.

Many authors have presented the derivation of the equations for orbital mechanics. Many referencing [1] which we shall also benefit from. See the work of [2],[6], and [4] as well as articles on Wikipedia. Also [4] is freely available on the Internet. He references [1]. The Course Notes in [7] is a great reference (also freely available). There is a reason we have written this chapter, so a comprehensive, connected derivation is provided with no ambiguities.

Of course, Kepler, Newton, and Leibniz are responsible for the equations and laws of orbital mechanics but much credit is also due to Tycho Brahe's astronomical observations as well as the Astronomers throughout history going back to the Babolonians. See [1] for great historical notes on Kepler.

In this paper, we derive the equations of motion for the two body system and provide all the details.

In Figure 4 for the two body system the only force on mass m is the gravitational force between mass m and mass M which is:

$$F_m = mMG\frac{\mathbf{r}}{r^3} \tag{16}$$

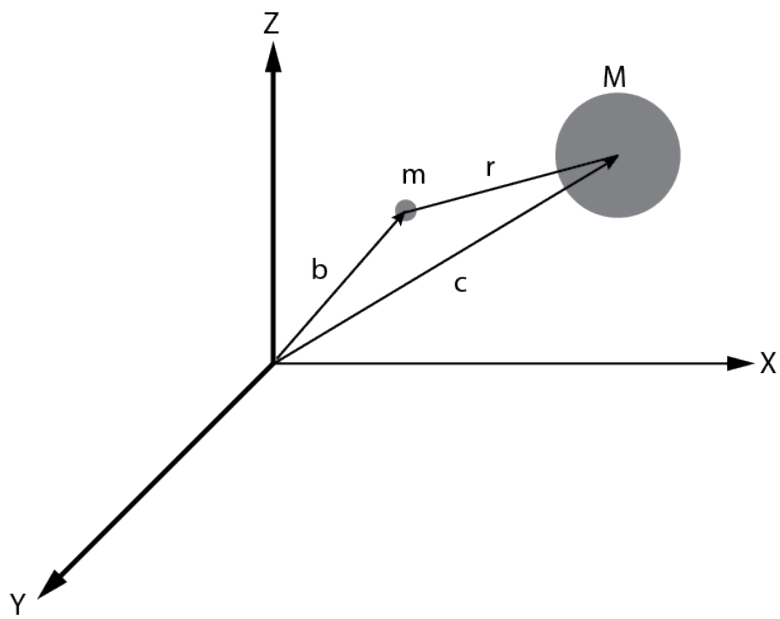

Figure 4: Two Body System

Note that the unit vector along $\mathbf{r}$ is $\frac{\mathbf{r}}{r}$. The gravitational force is inversely related to r^2.

The force on M is:

$$F_M = -mMG\frac{\mathbf{r}}{r^3} \tag{17}$$

Now, in the inertial frame.

$$m\ddot{\mathbf{b}} = mMG\frac{\mathbf{r}}{r^3} \tag{18}$$

and

$$m\ddot{\mathbf{c}} = -mMG\frac{\mathbf{r}}{r^3} \tag{19}$$

The vector $\mathbf{r}$ is:

$$\mathbf{r} = \mathbf{b} - \mathbf{c} \tag{20}$$

So,

$$\ddot{\mathbf{r}} = \ddot{\mathbf{b}} - \ddot{\mathbf{c}} \tag{21}$$

Therefore, subtracting (19) from (18) we obtain:

$$\ddot{\mathbf{r}} = -G(M - m)\frac{\mathbf{r}}{r^3} \tag{22}$$

If we set the origin of the Inertial Frame at the center of the assumed spherially symmetric and massive object with mass M and assuming $m \ll M$, then,

$$\ddot{\mathbf{r}} + GM\frac{\mathbf{r}}{r^3} = 0 \tag{23}$$

3.1 Conservation of Energy

The following derivation follows [2] and [6]. The conservation of energy can also be found in [4] which references [1].

Let $\mu = GM$. Then (23) becomes:

$$\ddot{\mathbf{r}} + \mu\frac{\mathbf{r}}{r^3} = 0 \tag{24}$$

Form the dot product of (24) with $\dot{\mathbf{r}}$,

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \mu\frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3} = 0 \tag{25}$$

Now,

$$\frac{d(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})}{dt} = 2\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} \tag{26}$$

and

$$\frac{d(\mathbf{r} \cdot \mathbf{r})}{dt} = 2\mathbf{r} \cdot \dot{\mathbf{r}} \tag{27}$$

So (25) can be written:

$$\frac{d(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})}{dt} + \frac{\mu}{r^3} \frac{d(\mathbf{r} \cdot \mathbf{r})}{dt} = 0 \tag{28}$$

Integrating (29) with respect to time t we obtain:

$$\dot{r}^2 + \frac{\mu}{r} = \epsilon \tag{29}$$

From [2], here $\dot{r}^2$ is the specific kinetic energy and $\frac{\mu}{r}$ is the specific potential energy of the object with mass m. Also see [4].

So as the object with mass m gains in kinetic energy (speed) its potential energy decreases (distance to mass M). Also as its potential enegy inceases (distance to mass M increases) its speed decreases. This will be examined in detail in a later chapter. Note that the object with mass M is fixed in location (the assumption that $m \ll M$).

3.2 Conservation of Angular Momentum

We know that angular momentum is ralated to $\mathbf{r} \times \dot{\mathbf{r}}$ so it makes sense to cross multiply (24) by $\mathbf{r}$.

$$\mathbf{r} \times \ddot{\mathbf{r}} + \mu \mathbf{r} \times \frac{\mathbf{r}}{r^3} = 0 \tag{30}$$

Thus,

$$\mathbf{r} \times \ddot{\mathbf{r}} = 0 \tag{31}$$

Since $\mathbf{r} \times \mathbf{r} = 0$.
However,

$$\frac{d(\mathbf{r} \times \dot{\mathbf{r}})}{dt} = \mathbf{r} \times \ddot{\mathbf{r}} + \dot{\mathbf{r}} \times \dot{\mathbf{r}} \tag{32}$$

Note that $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$.
Which leads to,

$$\mathbf{r} \times \ddot{\mathbf{r}} = \frac{d(\mathbf{r} \times \dot{\mathbf{r}})}{dt} \tag{33}$$

So,

$$\frac{d(\mathbf{r} \times \dot{\mathbf{r}})}{dt} = 0 \tag{34}$$

Or,

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{H} \tag{35}$$

where $\mathbf{H}$ is a constant vector. This means that $\mathbf{r}$ and $\dot{\mathbf{r}}$ are in the same plane throughout the motion of the object with mass m.

Now, following [2], cross multiplying (24) by $\mathbf{H}$,

$$\mathbf{H} \times \ddot{\mathbf{r}} + \mathbf{H} \times \mu \frac{\mathbf{r}}{r^3} = 0 \tag{36}$$

Or,

$$\ddot{\mathbf{r}} \times \mathbf{H} = \mu \frac{\mathbf{H} \times \mathbf{r}}{r^3} \tag{37}$$

Now,

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{H})}{dt} = \ddot{\mathbf{r}} \times \mathbf{H} + \mathbf{r} \times \dot{\mathbf{H}} \tag{38}$$

But $\mathbf{H}$ is a constant vector so $\dot{\mathbf{H}}$=0.

Finally,

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{H})}{dt} = \mu \frac{\mathbf{H} \times \mathbf{r}}{r^3} \tag{39}$$

Now, based on (35) :

$$\mathbf{H} \times \mathbf{r} = (\mathbf{r} \times \mathbf{v}) \times \mathbf{r} \tag{40}$$

Where $\mathbf{H} = \mathbf{v} \times \mathbf{r}$ is the angular momemntum.

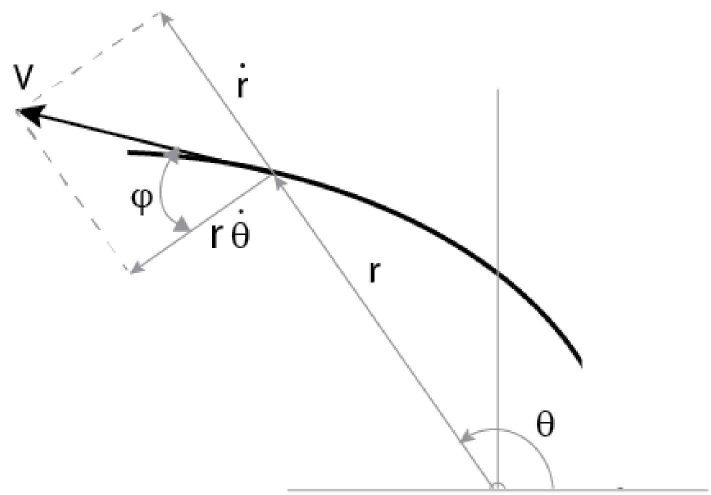

Figure 5: Velocity Vector

We will use the vector relationship:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{41}$$

So that

$$-(\mathbf{r} \times \mathbf{v}) \times \mathbf{r} = \mathbf{r} \times (\mathbf{r} \times \mathbf{v}) = (\mathbf{r} \cdot \mathbf{v})\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\mathbf{v} \tag{42}$$

Lets focus on the equation:

$$(\mathbf{r} \cdot \mathbf{v})\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\mathbf{v} \tag{43}$$

With reference to Figure 5, we will examine the term $\mathbf{r} \cdot \mathbf{v}$.

$$\mathbf{r} \cdot \mathbf{v} = rv\cos(\phi) \tag{44}$$

Now from Figure 5,

$$\dot{r} = v\cos(\phi) \tag{45}$$

Elliminate $\cos(\phi)$ from (44) and (45) we get,

$$\mathbf{r} \cdot \mathbf{v} = r\dot{r} \tag{46}$$

The above procedure for obtaining (46) was based on [6].
Thus we can write,

$$-(\mathbf{r} \times \mathbf{v}) \times \mathbf{r} = (\mathbf{r} \cdot \mathbf{v})\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\mathbf{v} = r\dot{r}\mathbf{r} - r^2\mathbf{v} \tag{47}$$

For reference we show equation (39)

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{H})}{dt} = \mu \frac{\mathbf{H} \times \mathbf{r}}{r^3} \tag{48}$$

Which we can write, based on (47), as:

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{H})}{dt} = -\mu \frac{r\dot{r}\mathbf{r} - r^2\mathbf{v}}{r^3} \tag{49}$$

Or,

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{H})}{dt} = -\mu(\frac{\dot{r}\mathbf{r}}{r^2} - \frac{\mathbf{v}}{r}) \tag{50}$$

Note that $\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2}$. Therefore we recognize that,

$$\frac{d}{dt}(\frac{\mathbf{r}}{r}) = -\frac{\dot{r}\mathbf{r}}{r^2} + \frac{\dot{\mathbf{v}}}{r} \tag{51}$$

Note that $\mathbf{v} = \dot{\mathbf{r}}$, see [5] , and we must distinguish between $\dot{\mathbf{r}}$ and $\dot{r}$.
Finally,

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{H})}{dt} = \frac{d}{dt}(\frac{\mathbf{r}}{r}) \tag{52}$$

Integrating both sides of (52) we obtain:

$$\dot{\mathbf{r}} \times \mathbf{H} = \frac{\mathbf{r}}{r} + \mathbf{B} \tag{53}$$

The vector $\mathbf{B}$ is a constant of integration.
Following [6] and [2], dot multiply both sides of (53) by $\mathbf{r}$

$$\mathbf{r} \cdot \dot{\mathbf{r}} \times \mathbf{H} = \mathbf{r} \cdot \frac{\mathbf{r}}{r} + \mathbf{r} \cdot \mathbf{B} \tag{54}$$

Following [6], we use the following identity:

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} \tag{55}$$

So that,

$$\mathbf{r} \cdot \dot{\mathbf{r}} \times \mathbf{H} = \mathbf{r} \times \dot{\mathbf{r}} \cdot \mathbf{H} = \mathbf{H} \cdot \mathbf{H} = H^2 \tag{56}$$

Now,

$$\mathbf{r} \cdot \frac{\mathbf{r}}{r} + \mathbf{r} \cdot \mathbf{B} = r + rBcos(\psi) \tag{57}$$

Which leads to, based on (54),

$$H^2 = \mu r + \mu Bcos(\psi) \tag{58}$$

Finally,

$$r = \frac{\frac{H^2}{\mu}}{1 + \frac{B}{\mu}\cos(\psi)} \tag{59}$$

See Chaper 1 for the mathematical treatment of this equation for the case that $e = \frac{B}{\mu} < 1$ which is an Ellipses. See Figure 2.

In general (59) describes conic sections. For conic sections with various mathematical representations and animations, see [3].

A comment of the angle between **B** and **r**. There is every reason to assume that this angle can be taken as θ as in Figure 5 especially since the vector **B** is a constant of integration.

3.3 Orbit Equation

Define,

$$p = \frac{H^2}{\mu} \tag{60}$$

p is called the "semi parameter" Also, define,

$$e = \frac{B}{\mu} \tag{61}$$

e is called the eccentricty.
So (59) can be written as

$$r = \frac{p}{1 + e\cos(\theta)} \tag{62}$$

In the following chapters θ is called the True Anomaly. In a later chapter we will show a graph using Mathematica® code where the velocity vector is shown as a function of the True Anomaly θ.

3.4 Velocity

Since the angular momentum is constant:

$$H = r_a v_a \tag{63}$$

$$H = r_p v_p \tag{64}$$

So the Energy is from (29) substituting for v_p (we are using E for energy)

$$E = \frac{v^2}{2} - \frac{\mu}{r} = \frac{H^2}{2r_p^2} - \frac{\mu}{r_p} \tag{65}$$

At $\theta = 0$, we have $r_p = \frac{p}{1+e}$. But $p = a(1-e^2)$ so

$$r_p = \frac{a(1-e^2)}{1+e} = a(1-e) \tag{66}$$

From $r_p = a(1-e)$ and from (60) $H = \sqrt{\mu p} = \sqrt{\mu a(1-e^2)}$ so,

$$E = \frac{\mu a(1-e^2)}{2a^2(1-e)^2} - \frac{\mu}{a(1-e)} \tag{67}$$

For $e \neq 0$, i.e. not a parabolic path, this reduces to,

$$E = -\frac{\mu}{2a} \tag{68}$$

Since E is constant, then from (65) substituing for E and rearranging,

$$v^2 = \mu(\frac{2}{r} - \frac{1}{a}) \tag{69}$$

3.5 Elliptic Orbit Period

Recall that the transverse component of the velocity vector was $v\cos(\phi)$ from Figure 5. So it is this component that will lead us to compute the Period as it relates to the rate at which the object is moving. Following [6] , we can express the transverse component of the velocity vector from Figure 5 as $r\dot{\theta}$. Now the constant angular momentum is $H = r\dot{\theta}r$. Note that the the $\dot{r}$ component does not contribute to H in $H = \mathbf{r} \times \mathbf{v}$. Or,

$$H = r^2 \frac{d\theta}{dt} \tag{70}$$

Which can be rearranged as,

$$dt = \frac{r^2 d\theta}{H} \tag{71}$$

Now this is interesting. The area swept by $d\theta$ is $dA = \frac{1}{2}r(rd\theta) = \frac{1}{2}r^2 d\theta$. So if we substitute for $r^2\theta$ in (71) we obtain:

$$dt = 2\frac{dA}{H} \tag{72}$$

Integrating over θ from 0 to 2π gives,

$$P = T = 2\frac{A_{ellipse}}{H} \tag{73}$$

The area of the Ellipse (See Figure 1), is

$$A_{ellipse} = \pi ab \tag{74}$$

Substituting into (75),

$$P = T = 2\frac{\pi ab}{H} \tag{75}$$

Now $p = \frac{H^2}{\mu}$. See equation (60). So $H = \sqrt{p\mu}$. Also $p = a(1 - e^2)$ so,

$$P = T = 2\frac{\pi ab}{\sqrt{p\mu}} = 2\frac{\pi ab}{\sqrt{a(1-e^2)\mu}} \tag{76}$$

Now $b = a\sqrt{1-e^2}$. So,

$$P = 2\frac{\pi a^2\sqrt{1-e^2}}{\sqrt{a(1-e^2)\mu}} \tag{77}$$

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{78}$$

3.6 Summary of Equations for Position and Velocity

Constants

$$\boxed{\begin{aligned} \mu &= GM && (79a)\\ G &= 6.67430 \times 10^{-11} N - m^2 \cdot kg^{-2} && (79b)\\ M_{Earth} &= 5.97221024kg && (79c)\\ \mu_{Earth} &= 3.986004415 \times 10^5 \frac{km^3}{s^2} && (79d) \end{aligned}}$$

Equations

$$\boxed{\begin{aligned} p &= \frac{H^2}{\mu} && (80a)\\ p &= a(1-e^2) && (80b)\\ r &= \frac{p}{1+e\cos(\theta)} && (80c)\\ v &= \sqrt{\mu(\frac{2}{r}-\frac{1}{a})} && (80d)\\ P &= 2\pi\sqrt{\frac{a^3}{\mu}} && (80e) \end{aligned}}$$

Chapter 4

Earth and Jupiter Orbits

4.1 An Earth Orbit

The parameters used for the Satellite are from [6] and are summarized below:

$$\boxed{\begin{aligned} \mu &= 3.986004415 * 10^5 && (81a)\\ e &= 0.6 && (81b)\\ p &= 20,410\ km && (81c)\\ a &= 31890\ km && (81d)\\ b &= 25512\ km && (81e)\\ r_p &= 12756\ km && (81f)\\ r_a &= 51024\ km && (81g)\\ c &= 19134\ km && (81h)\\ EarthRadius &= 6371\ km && (81i) \end{aligned}}$$

Constants

$$\boxed{\begin{aligned} \mu &= GM && (82a)\\ G &= 6.67430 \times 10^{-11} N - m^2 \cdot kg^{-2} && (82b)\\ M_{Earth} &= 5.9722 \times 10^{24} kg && (82c)\\ \mu_{Earth} &= 3.986004415 \times 10^5 \frac{km^3}{s^2} && (82d) \end{aligned}}$$

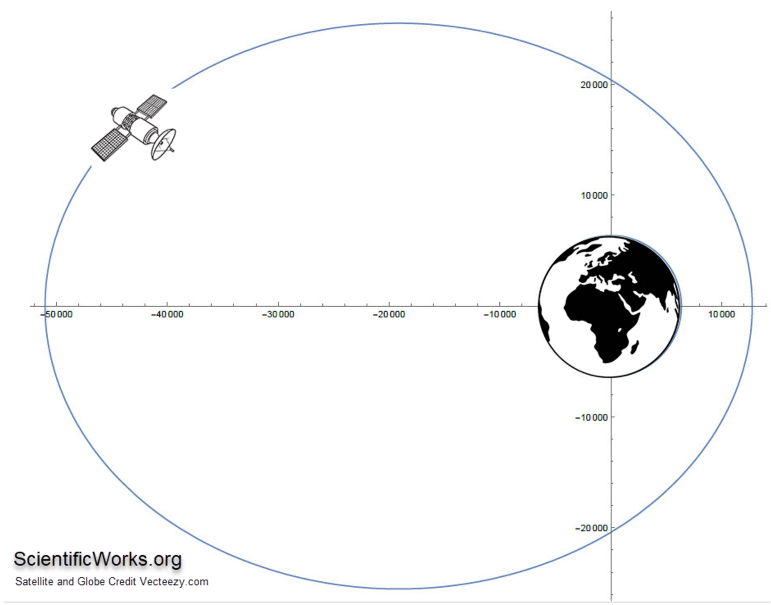

Figure 6: Elliptic Earth Orbit

Table 1: Space Station Calculated Orbit Parameters

Item	Parameter	Value	Units
1	$mass_{Earth}$ Constant	5.97×10^{24}	kg
2	μ Constant	3.98603×10^{5}	$km^3 s^{-2}$
3	Period(P) Specified	0.064513889	Days
4	Eccentricity(e) Specified	0.0008051	
5	a	6796.9754	km
6	p	6796.9710	km
7	b	6796.9732	km
8	c	5470.427305	km
9	r_p	6791.5032	km
10	r_a	6800.188	km
11	Velocity at r=r_p	7665.38	m/s
12	Velocity at $r = r_p$	27604.555	km/H

4.2 Space Station

The specified parameter for the Space Station is the Period which is $P = 92.9$ minutes and an eccentricity $e = 0.0008051$. Below we calculate the orbital parameters.

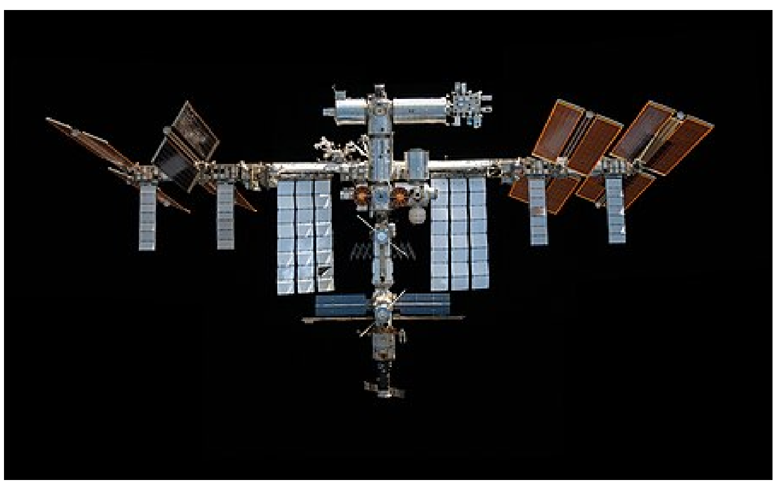

Figure 7: Space Station Wikipedia

Note that the height of the space station at r_p is:

$$h_{spacestation} = r_p - Radius_{Earth} = 6791.5 - 6378 = 420.5\ km \tag{83}$$

Figure 8 shows the computed and graphed orbit with Earth using Mathematica® code.

See the section on Juno's Jupiters Orbit for the Mathematica® code.

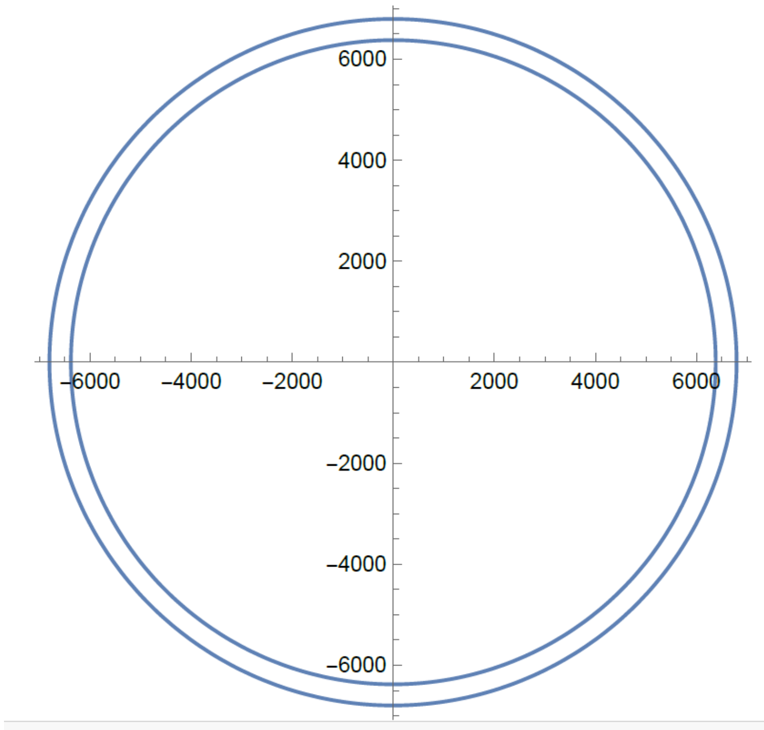

Figure 8: Spacestation Mathematica® Orbit Calculation, Units in Km

4.3 Juno Spacecraft Jupiter Orbit

Juno is a NASA spacecraft operated by JPL. It is orbiting Jupiter. In this section we will calculate Juno's Jupiter Orbit. The orbit is also plotted with Mathematica® code, as well as Jupiter itself. The velocity vector of the orbit is also plotted with Mathematica® code. Figure 9 shows the spacecraft.

Figure 9: Juno Spacecraft Wikipedia

Constants

$$\boxed{\begin{aligned} \mu &= GM && (84a)\\ G &= 6.67430 \times 10^{-11} N - m^2 \cdot kg^{-2} && (84b)\\ M_{Jupiter} &= 1.898 \times 10^{27} kg && (84c)\\ \mu_{Jupiter} &= 1.26678 \times 10^{17} \frac{km^3}{s^2} && (84d)\\ Radius &= 71,600 km && (84e) \end{aligned}}$$

The period for Juno's orbit is 11.07 days. So we can calculate a based on (85).

$$a^3 = \frac{P^2 \mu}{(2\pi)^2} \tag{85}$$

$$a = 1,431,819.63\ km \tag{86}$$

Table 2: Juno Jupiter Orbit Calculated Parameters

Item	Parameter	Value	Units
1	$mass_{Jupiter}$ Constant	1.9×10^{27}	kg
2	μ Constant	1.26678×10^{17}	$km^3 s^{-2}$
3	Period(P) Specified	11.07	Days
4	Eccentricity(e) Specified	0.68	
5	a	1,431,819.6	km
6	p	769,746.2	km
7	b	1,049,827.5	km
8	c	973,637.35	km
9	r_p	458,183	km
10	r_a	2,405,457	km
11	Velocity at r=r_p	21552	m/s
12	Velocity at $r = r_p$	77,587,000	km/H

The Orbits eccentricity is:

$$e = 0.68 \tag{87}$$

So $p = a(1 - e^2)$ is:

$$p = 769746km \tag{88}$$

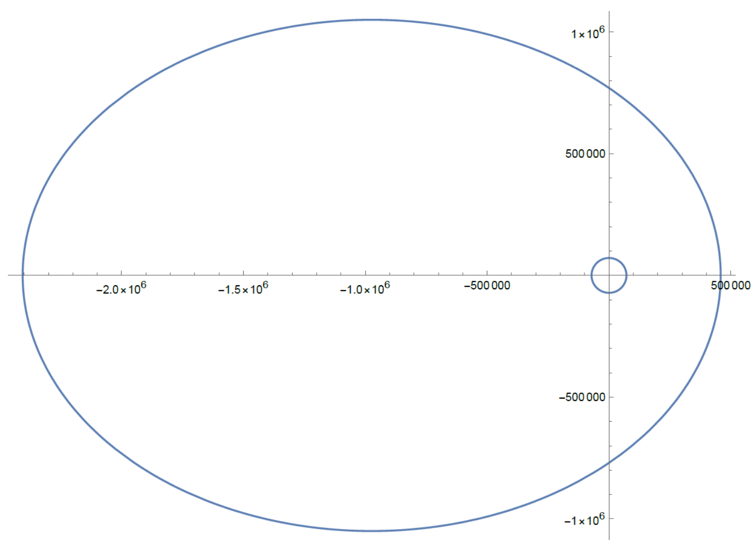

Figure 10: Juno Jupiter Orbit

Mathematica® Code:

```
mu   = 3.986004415*10^5
e  = 0.68
p  = 769843.2
a  = 1432000
b  = 1049959.743
rp  = 2405760
ra  = 2405760
c  = 973760
jupiterRadius = 71600
period = 2*Pi*Sqrt[a*a*a/mu]/3600
Show[PolarPlot[p/(1 + e*Cos[nu]), \{nu, 1, 2.5*Pi\}],
PolarPlot[jupiterRadius, \{nu, 1, 2.5*Pi\}]]
Plot[Sqrt[mu*(2/(p/(1 + e*Cos[nu]) - 1/a))],
 \{nu, 1, 2.5*Pi\}]
```

We also plot the velocity vector. Links to the code is provided in a later Chapter.

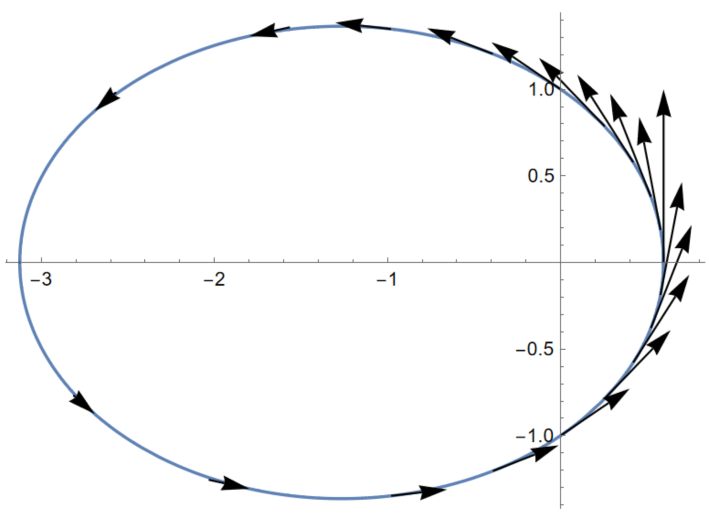

Figure 11: Juno Jupiter Orbit Velocity Vector

Chapter 5

Orbit Position as a Function of Time for Elliptic Orbits

This chapter covers aspects of predicting the position of an object in orbit by specifying the time. The position of, for example a satellite, as it orbits, given the angle θ is straight forward. However, predicting the position of the satellite by specifiying time is a whole different matter.

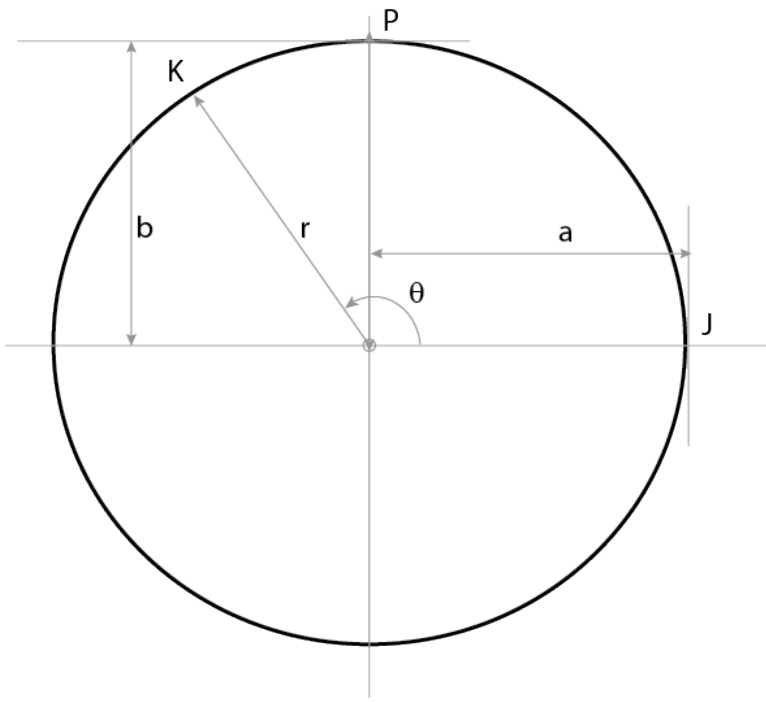

Figure 12: Circular Orbit

Consider a circular orbit as shown in Figure 12. In this case, if we specify the position with the True Analmoly θ the position at K is known. Now, if we start out at the point where $\theta = 0$, J in Figure 12, and call that time $t = 0$, then, for example a satellite at K will return to position J at $\theta = 0$ after T seconds where T is the orbit period.

The orbit period T is calculated in (89), and is true for both Elliptic and Circular Orbits. In the case of a Circular Orbit $r = a$ and $e = 0$.

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{89}$$

So to predict the position of the Satellite at K corresponding to a specified time t we need to determined θ . But for a Circular Orbit,

$$\theta = t\frac{2\pi}{T} \tag{90}$$

Noting that for $t = T$ we return back to J with $\theta = 2\pi$.

Also since the velocity,

$$v = \sqrt{\mu(\frac{2}{r} - \frac{1}{a})} \tag{91}$$

and $r = a$ the velocity is uniform throught out the Circular Orbit. So (90) holds.

For non circular orbits, the velocity is changing with position. In the following we will outline how to predict position, with specified time.

Now define n, the mean motion, as,

$$n = \sqrt{\frac{\mu}{a^3}} \tag{92}$$

So base on (89),

$$n = \frac{2\pi}{P} \tag{93}$$

Figures 13 and 14 show the auxillary circle and the eccentric anomaly E for two diffrenet positions θ.

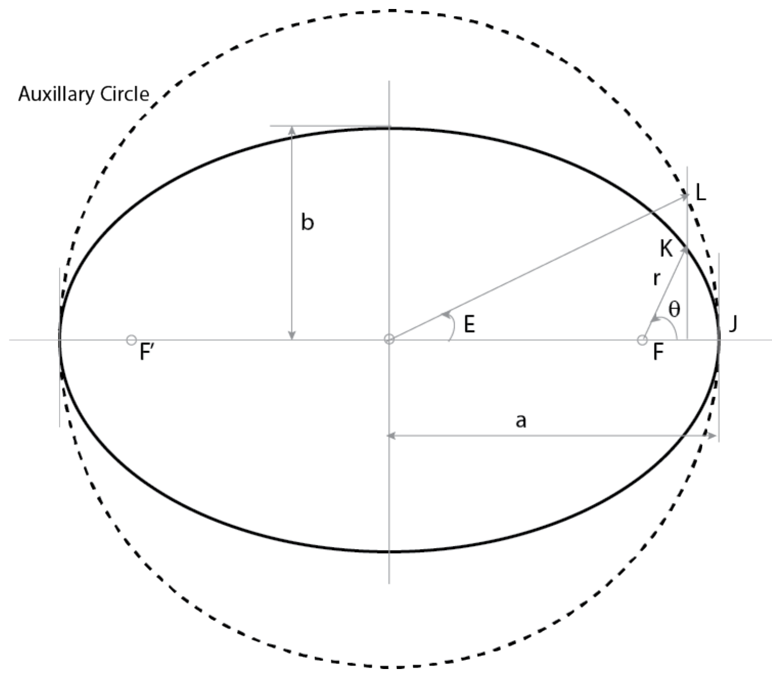

Figure 13: Elliptic Circular Orbit With Auxillary Circle and Showing Eccentric Anamoly, E

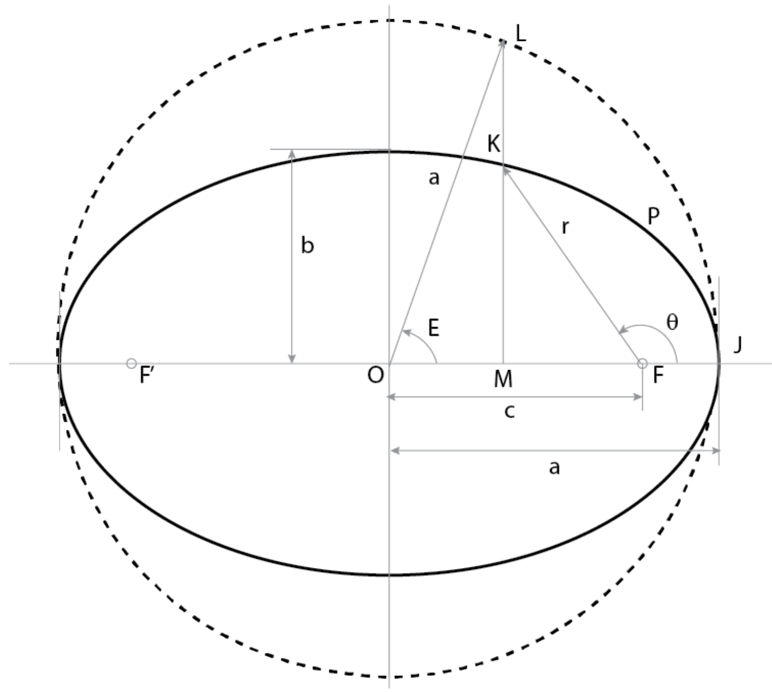

Figure 14: Elliptic Circular Orbit With Auxillary Circle and Showing Eccentric Anamoly, E Another Location

In the developments to come, we will need to express the eccentric anomaly E in terms of θ and vice versa. First lets relate E to θ.

In Figure 14,

$$\cos(E) = \frac{OM}{OL} = \frac{OF - MF}{OL} = \frac{c + r\cos(\theta)}{a} \tag{94}$$

Or in terms of the eccentricity $e = \frac{c}{a}$, and noting that $r = \frac{p}{1+e\cos(\theta)}$ and that $p = a(1-e^2)$,

$$\cos(E) = \frac{e + \cos(\theta)}{1 + e\cos(\theta)} \tag{95}$$

5.1 Analytical Method

See [1] and [7] for the Analytic Method for determining position based on time of flight. We shall derive the analytical equation by following [1]. We use our notation.

Recall from equation (71)

$$dt = \frac{r^2 d\theta}{H} \tag{96}$$

Integrating,

$$\int_T^t H dt = \int_0^\theta r^2 d\theta \tag{97}$$

Or,

$$(t - T)H = \int_0^\theta \frac{p^2 d\theta}{[1 + e\cos(\theta)]^2} \tag{98}$$

Now a change of variables following [1] to the eccentric anomaly E is in order. Recall that from (99),

$$\cos(E) = \frac{e + cos(\theta)}{1 + e\cos(\theta)} \tag{99}$$

the following relations are established between θ and E:

$$\cos(\theta) = \frac{e - cos(E)}{e\cos(E) - 1} \tag{100}$$

and,

$$\sin(\theta) = \frac{a\sqrt{1 - e^2}}{r}\sin(E) \tag{101}$$

Substituting for $\cos(\theta)$ using (100) noting that $r = \frac{p}{1+e\cos(\theta)}$

Now to obtain the relationship between E and r just like we have the relationship between r and θ, we proceed as outlined in [6].

We start from (94) which we show below:

$$\cos(E) = \frac{c + r\cos(\theta)}{a} \tag{102}$$

Then,

$$r = \frac{a\cos(E) - c}{\cos(\theta)} \tag{103}$$

$$r = \frac{a\cos(E) - c}{\frac{\cos(E) - e}{1 - e\cos(E)}} = \frac{(a\cos(E) - c)[1 - e\cos(E)]}{\cos(E) - e} \tag{104}$$

$$r = \frac{a(\cos(E) - e)[1 - e\cos(E)]}{\cos(E) - e} \tag{105}$$

Finally,

$$r = a[1 - e\cos(E)] \tag{106}$$

Differentiating (100) we obtain,

$$d\theta = \frac{\sin(E)[1 + e\cos(\theta)]}{\sin(\theta)[1 - e\cos(E)]} dE = \frac{\sin(E)(\frac{p}{r})}{\sin(\theta)\frac{r}{a}} dE \tag{107}$$

$$d\theta = \frac{a\sqrt{1 - e^2}}{r} dE \tag{108}$$

Where we substituted for $\sin(\theta)$ using (101) in (107).

Then,

$$(t - T)H = \frac{p}{\sqrt{1 - e^2}} \int_0^E r dE \tag{109}$$

$$(t - T)H = \frac{pa}{\sqrt{1 - e^2}} \int_0^E [1 - e\cos(E)] dE \tag{110}$$

$$(t - T)H = \frac{pa}{\sqrt{1 - e^2}} [E - e\sin(E)] \tag{111}$$

Since $H = \sqrt{\mu p}$,

$$(t - T) = \frac{a^3}{\mu} [E - e\sin(E)] \tag{112}$$

This equation is referred as Kepler's equation. This equation can be solved for E given t using Newton-Raphson successive Approximation. There is no closed-form solution (never dispair!). In the next section we present Mathematica® code (Open Source, GPL3) for solving for Position as a Function of Time. The key equations are summarized below.

$$\boxed{\begin{aligned}
(t-T) &= \frac{a^3}{\mu}[E - e\sin(E)] && (113a)\\
\cos(\theta) &= \frac{e - cos(E)}{e\cos(E) - 1} && (113b)\\
\sin(\theta) &= \frac{a\sqrt{1-e^2}}{r}\sin(E) && (113c)\\
\tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} && (113d)\\
\theta &= \tan^{-1}(\theta) && (113e)
\end{aligned}}$$

5.2 Mathematica® Code Orbit Position as a Function of Time for Elliptic Orbits

For the latest updates to the Mathematica® code visit:

```
https://scientificworks.org/orbital_mechanics.html
```

In the code, the following parameters were used [6]:

$$\boxed{\begin{aligned} \mu &= 3.986004415 * 10^5 && (114a)\\ e &= 0.6 && (114b)\\ p &= 20,410\ km && (114c)\\ a &= 31890\ km && (114d)\\ b &= 25512\ km && (114e)\\ r_p &= 12756\ km && (114f)\\ r_a &= 51024\ km && (114g)\\ c &= 19134\ km && (114h)\\ EarthRadius &= 6371\ km && (114i) \end{aligned}}$$

Constants

$$\boxed{\begin{aligned} \mu &= GM && (115a)\\ G &= 6.67430 \times 10^{-11} N - m^2 \cdot kg^{-2} && (115b)\\ M_{Earth} &= 5.9722 \times 10^{24} kg && (115c)\\ \mu_{Earth} &= 3.986004415 \times 10^5 \frac{km^3}{s^2} && (115d) \end{aligned}}$$

The results are shown in Figures 15, 16, and 17. Note that in both Figures 16 and 17 the velocity decreases as time increases from (t0) Perigee to Apogee t=T.

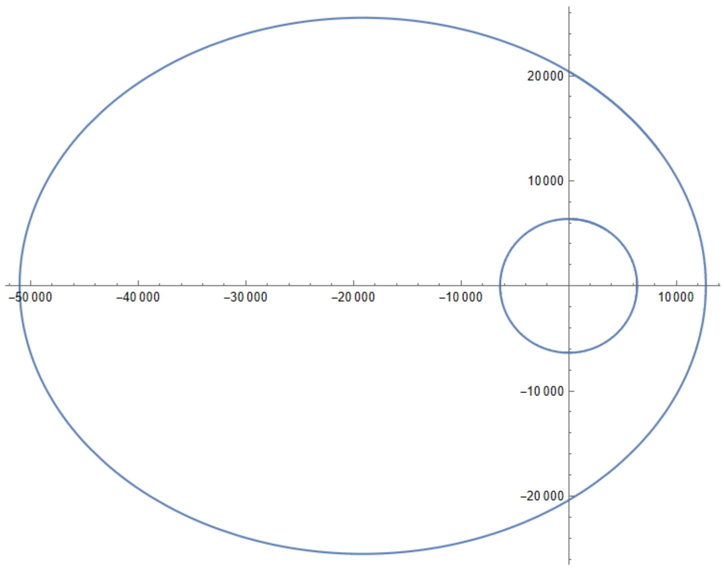

Figure 15: Elliptic Orbit Mathematica® Code

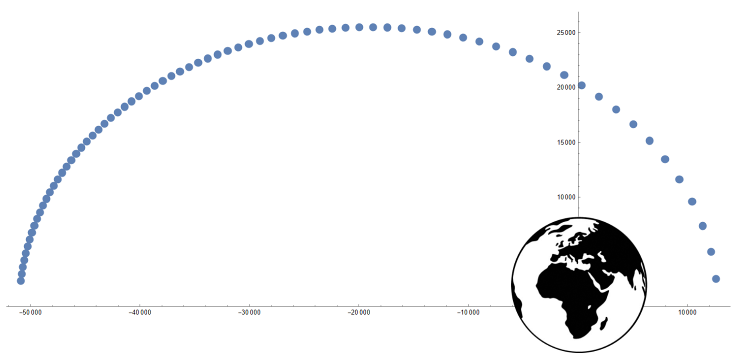

Figure 16: Position θ True Anomaly Orbit as Time is Varied for 7.5 Hours with Period = 15.74 Hours Mathematica® Code

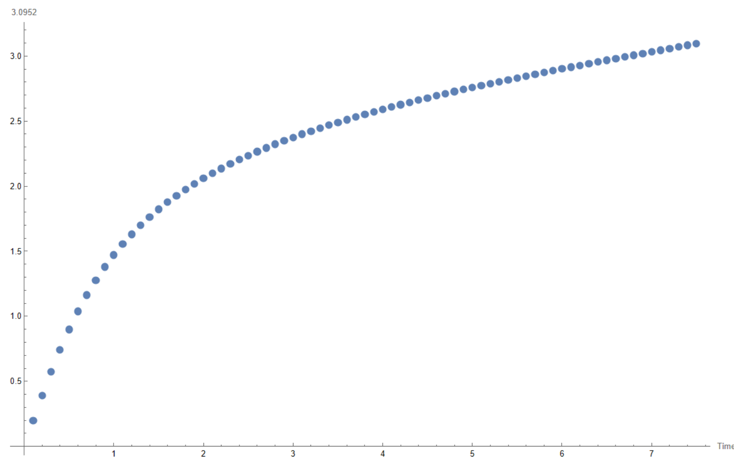

Figure 17: Position θ True Anomaly as Time is Varied from 0 to 7.5 Hours in 0.1 Hour Steps with Period = 15.74 Hours Mathematica® Code

Mathematica® Code Orbit Position as a Function of Time for Elliptic Orbits

```
(* Copyright (C)2024 Sasan Ardalan          *)
(* This program is free software:you can redistribute it  *)
(* and/or modify   it under the terms *)
(* of the GNU General Public License *)
(* as published by *)
(* the Free Software Foundation,either version 3 of the License, *)
(* or at your option, any later version. *)
(* This program is distributed      *)
(* in the hope that it will be useful,but WITHOUT ANY WARRANT  *)
(* without even the implied warranty of   *)
(* MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE.   *)
(*See the  GNU General Public License for more details. *)
(*You should have              *)
(* received a copy of the GNU General Public License  *)
(* along with this program.If not,see.*)


(* Orbit Position as a Function of Time for Elliptic Orbits  *)
(* Orbital Mechanics                                               *)
(* Author:Sasan Ardalan                                         *)
(* ScientificWorks.org                                             *)
(* Date:September 10,2024                                     *)

(* Parameters from:                                           *)
(* Fundamentals of Astrodynamics and Applications,*)
(*  4th Edition, Vallado                                        *)
(* Earth Orbit                                                     *)

mu = 3.986004415*10^5;

e = 0.6;
p = 20410;
a = 31890;
b = 25512;
rp = 12756;
ra = 51024;
c = 19134;
earthRadius = 6371;

Print["Period is:"]
period = 2*Pi*Sqrt[a*a*a/mu]/3600

Show[PolarPlot[p/(1 + e*Cos[theta]), {theta, 1, 2.5*Pi}],
 PolarPlot[earthRadius, {theta, 1, 2.5*Pi}]]
```

```
SinE[theta_, e_] := Sqrt[1 - e*e] Sin[theta]/(1 + e*Cos[theta])
CosE[theta_, e_] := (e + Cos[theta])/(1 + e*Cos[theta])
SinTheta[Ev_, e_] := (Sqrt[1 - e*e]*Sin[Ev])/(1 - e*Cos[Ev])
CosTheta[Ev_, e_] := (Cos[Ev] - e)/(1 - e*Cos[Ev])

P = 2*Pi*Sqrt[a*a*a/mu]
Phours = P/3600
n = 2*Pi/P

satPosTime[th_, p_, e_, n_] :=
 Module[{t, t0, M0, EvRoot, T, M, Ev, thetaval}, t = th*3600; t0 = 0;
  T = 0; M0 = n*(t0 - T); M = n*(t - t0) + M0;
  EvRoot = Ev /. FindRoot[M == Ev - e*Sin[Ev], {Ev, M0}];
  sinTheta = SinTheta[EvRoot, e];
  cosTheta = CosTheta[EvRoot, e];

  thetaval = ArcTan[sinTheta/cosTheta];
  thetaval = If[thetaval < 0, thetaval + Pi, thetaval];
    Return[thetaval]  ]

Print["Sat Positione Time:"]
satPosTime[0.1, p, e, n]*180/Pi

satpos[th_, p_, e_, n_] := p/(1 + e*Cos[satPosTime[th, p, e, n]]);

Print["Sat Radius Time:"]
th1 = 7
satPosTime[th1, p, e, n]*180/Pi
satpos[th1, p, e, n]
```

```
track = {, }
path = {, }
th1 = 0
delta = 0.1;
For[i = 0, i < 75, i++, th1 = th1 + delta;
         theta = satPosTime[th1, p, e, n];
          r = satpos[th1, p, e, n];
          velocity = Sqrt[mu*(2/r - 1/a)];
          (* track=Append[track,{theta,velocity}]; *)
          (*track=Append[track,{th1,velocity}];*)

          track = Append[track, {th1,  theta  }];

          path = Append[path, {theta, r}];

         (*Print[r,":",theta*180/Pi,":",velocity];*)
 ]
(*Print ["The Track"]
Print[track]*)
ListPlot[track, AxesLabel -> {Time, theta}]
(* Print ["The Path"]
Print[path] *)
ListPolarPlot[path]
```

Chapter 6

Plotting the Velocity Vector in Mathematica®

It is important to visualize the velocity vector as function of tue anomaly (θ). For this purpose Mathematica® code was written. For reference, Figure 18 shows the orbit with Earth. This is the same as in Figure 15. The results for the parameters in [6] are shown in Figures 19.

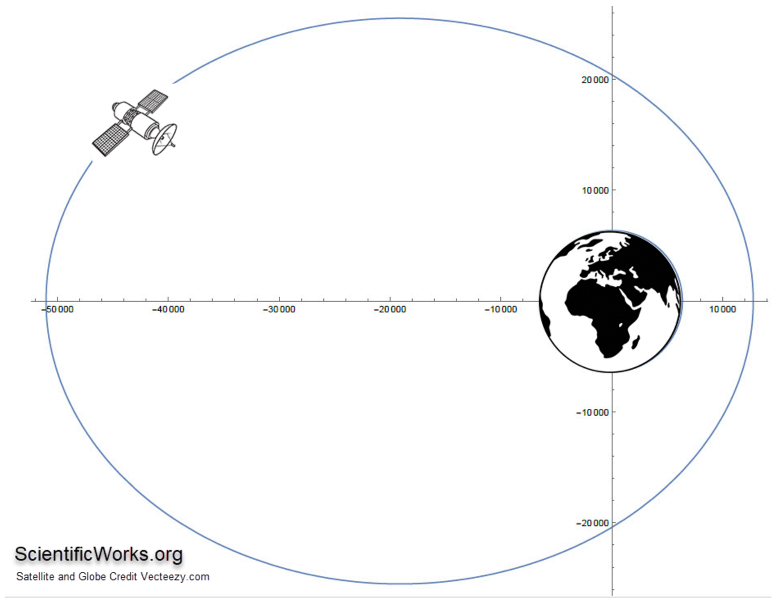

Figure 18: Elliptic Orbit Mathematica® Code

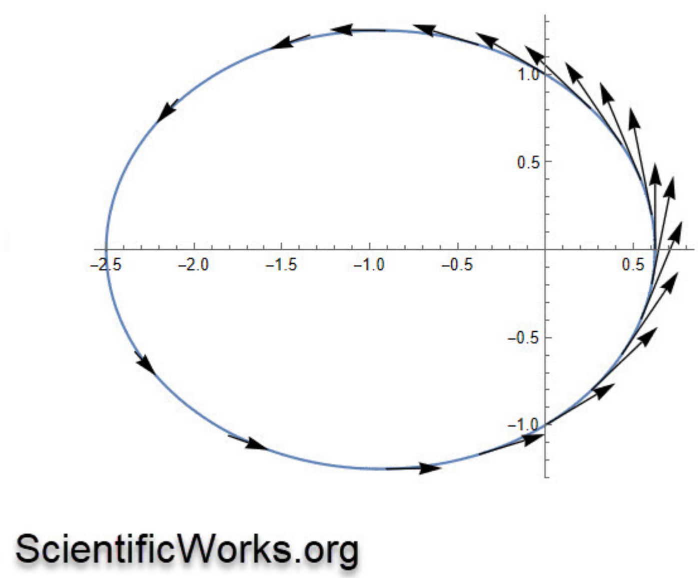

Figure 19: Velocity Vector Mathematica® Code

The Mathematica® code is provided in the following link:

```
https://scientificworks.org/orbital_mechanics_velocity_vector.html
```

The code is Open Source and Licensed under GPL3.

Bibliography

[1] R. R. Bate, Mueller D. D., and White J. E. In *Fundamentals of Astrodynamics*. Dover Publications, New York, 1971.

[2] Vladimir A. Chobotov. In *Orbital Mechanics, Third Edition*. The American Institute of Aeronautics and Astronautics, Inc., 2002.

[3] Wikipedia contributors. In *Conic Section*. Wikipedia, The Free Encyclopedia, 2024.

[4] Steven L. Rickman. In *Introduction to Orbital Mechanics and Spacecraft Attitudes for Thermal Engineers*. NASA Engineering and Safety Center, 2020.

[5] Jeremy Tatum. In *Two-dimensional polar coordinates*. LibreTexts Physics, 2024.

[6] David Vallado. In *Fundamentals of Astrodynamics and Applications, Fourth Edition*. Microcosm Press, 2003.

[7] David J. Westpfahl. In *Orbital Mechanics Course Notes*. David J. Westpfahl, Course Notes New Mexico Institute of Mining and Technology, 2011.

Chapter 7

GNU Free Documentation License

GNU Free Documentation License Version 1.3, 3 November 2008
Copyright ©2000, 2001, 2002, 2007, 2008 Free Software Foundation, Inc. <http://fsf.org/> Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed.

7.1 0. PREAMBLE

The purpose of this License is to make a manual, textbook, or other functional and useful document "free" in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or noncommercially. Secondarily, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.
This License is a kind of "copyleft", which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.
We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

7.2 1. APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein.

The "Document", below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as "you". You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law. A "Modified Version" of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A "Secondary Section" is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document's overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The "Invariant Sections" are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none. The "Cover Texts" are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A "Transparent" copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not "Transparent" is called "Opaque".

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD, and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.

The "Title Page" means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires

to appear in the title page. For works in formats which do not have any title page as such, "Title Page" means the text near the most prominent appearance of the work's title, preceding the beginning of the body of the text.
The "publisher" means any person or entity that distributes copies of the Document to the public.
A section "Entitled XYZ" means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as "Acknowledgements", "Dedications", "Endorsements", or "History".) To "Preserve the Title" of such a section when you modify the Document means that it remains a section "Entitled XYZ" according to this definition.
The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License.

7.3 2. VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or noncommercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section 3.
You may also lend copies, under the same conditions stated above, and you may publicly display copies.

7.4 3. COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document's license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.
If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and

continue the rest onto adjacent pages.
If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.
It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

7.5 4. MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections 2 and 3 above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:
• A. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.
• B. List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement.
• C. State on the Title page the name of the publisher of the Modified Version, as the publisher.
• D. Preserve all the copyright notices of the Document.
• E. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.
• F. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.
• G. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice.
• H. Include an unaltered copy of this License.
• I. Preserve the section Entitled "History", Preserve its Title, and add to it an

item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.

• J. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the "History" section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.

• K. For any section Entitled "Acknowledgements" or "Dedications", Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.

• L. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.

• M. Delete any section Entitled "Endorsements". Such a section may not be included in the Modified Version.

• N. Do not retitle any existing section to be Entitled "Endorsements" or to conflict in title with any Invariant Section.

• O. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version's license notice. These titles must be distinct from any other section titles.

You may add a section Entitled "Endorsements", provided it contains nothing but endorsements of your Modified Version by various parties?for example, statements of peer review or that the text has been approved by an organization as the authoritative definition of a standard. You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one. The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

7.6 5. COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section 4 above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.
The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work. In the combination, you must combine any sections Entitled "History" in the various original documents, forming one section Entitled "History"; likewise combine any sections Entitled "Acknowledgements", and any sections Entitled "Dedications". You must delete all sections Entitled "Endorsements".

7.7 6. COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects. You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

7.8 7. AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an "aggregate" if the copyright resulting from the compilation is not used to limit the legal rights of the compilation's users beyond what the individual works permit. When the Document is included in an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document. If the Cover Text requirement of section 3 is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document's Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate.

7.9 8. TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section 4. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail. If a section in the Document is Entitled "Acknowledgements", "Dedications", or "History", the requirement (section 4) to Preserve its Title (section 1) will typically require changing the actual title.

7.10 9. TERMINATION

You may not copy, modify, sublicense, or distribute the Document except as expressly provided under this License. Any attempt otherwise to copy, modify, sublicense, or distribute it is void, and will automatically terminate your rights under this License. However, if you cease all violation of this License, then your license from a particular copyright holder is reinstated (a) provisionally, unless and until the copyright holder explicitly and finally terminates your license, and (b) permanently, if the copyright holder fails to notify you of the violation by some reasonable means prior to 60 days after the cessation. Moreover, your license from a particular copyright holder is reinstated permanently if the copyright holder notifies you of the violation by some reasonable means, this is the first time you have received notice of violation of this License (for any work) from that copyright holder, and you cure the violation prior to 30 days after your receipt of the notice. Termination of your rights under this section does not terminate the licenses of parties who have received copies or rights from you under this License. If your rights have been terminated and not permanently reinstated, receipt of a copy of some or all of the same material does not give you any rights to use it.

7.11 10. FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See http://www.gnu.org/copyleft/. Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License "or any later version" applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation. If the Document specifies that a proxy can decide which future versions of this License can be used, that proxy's public statement of acceptance of a version permanently authorizes you to choose that version for the Document.

7.12 11. RELICENSING

?Massive Multiauthor Collaboration Site" (or "MMC Site") means any World Wide Web server that publishes copyrightable works and also provides prominent facilities for anybody to edit those works. A public wiki that anybody can edit is an example of such a server. A "Massive Multiauthor Collaboration" (or "MMC") contained in the site means any set of copyrightable works thus published on the MMC site. "CC-BY-SA" means the Creative Commons Attribution-Share Alike 3.0 license published by Creative Commons Corporation, a not-for-profit corporation with a principal place of business in San Francisco, California, as well as future copyleft versions of that license published by that same organization. "Incorporate" means to publish or republish a Document, in whole or in part, as part of another Document. An MMC is "eligible for relicensing" if it is licensed under this License, and if all works that were first published under this License somewhere other than this MMC, and subsequently incorporated in whole or in part into the MMC, (1) had no cover texts or invariant sections, and (2) were thus incorporated prior to November 1, 2008. The operator of an MMC Site may republish an MMC contained in the site under CC-BY-SA on the same site at any time before August 1, 2009, provided the MMC is eligible for relicensing.